AIAA 82-4117

Hamilton's Principle, Hamilton's Law—6^n Correct Formulations

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The standard formulation of Hamilton's principle requires the knowledge of the initial and end coordinates of a given dynamic system. Any attempt to use this principle for direct calculations without the variations of the initial and end coordinates vanishing will bring incorrect results. It seems that this is the reason for the appearance in the scientific literature of arguments about the fitness of Hamilton's principle, and more generally Hamilton's law, for direct calculations of dynamic systems. However, it will be shown in this paper that there exist six correct formulations of Hamilton's law for each degree of freedom of the dynamic system. These formulations also include Hamilton's principle. The use of all of the 6^n correct formulations yields correct results. The paper centers around 6^n correct formulations of Hamilton's law. Actually, as will be shown, there are more. The principle of virtual work, which is equivalent to Hamilton's law, has many possibilities because the coordinates and their variations can be regarded as being mutually independent.

Nomenclature

a_i	= unknown coefficients of the power series
E_f	= effective coefficient for error estimate
F1-F6	= formulations, six different combinations of vanishing variations
f_i	= path-dependent nonconservative forces
$q_i, \dot{q}_i$	= generalized coordinates: displacement, velocity
T	= kinetic energy
t	= time
t_0	= initial time
t_f	= final time
V	= potential energy
δ	= variation (virtual) operator
τ	= nondimensional time $= t/k$

Introduction

IN the years 1834 and 1835, William Rowan Hamilton[1,2] published two papers in which appeared his "Law of Varying Action" (Ref. 1, p. 253). A special case of this law is obtained "if the final and initial coordinates and the time be given" (Ref. 2, p. 99) and therefore do not vary. This last case is known in the scientific literature (see, for example, Ref. 3, p. 88) as "Hamilton's principle."

Hamilton's principle, written in a way which also includes nonconservative forces, is given in many textbooks and papers (see, for example, Ref. 3 or 4),

$$\delta\int_{t_0}^{t_f}(T-V)\,\mathrm{d}t+\int_{t_0}^{t_f}\Big(\sum_i f_i\delta q_i\Big)\mathrm{d}t=0 \tag{1}$$

where T is the kinetic energy of the system, V its potential energy, and f_i are path-dependent nonconservative forces which cannot be included in the variational operation.

Now, Hamilton's principle [Eq. (1)] must be accompanied with the requirement that "the final and initial coordinates and the time be given."[2] Attempts to use Eq. (1) for direct calculations of given dynamic systems without fulfilling this last necessary requirement have brought incorrect results and deep disappointments. This caused a number of scientists[5-11] to search for some variational principles which would include the initial conditions implicitly.

Received Oct. 28, 1980; revision received Sept. 15, 1981.

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Bailey[12-19] states that "in real life" the final position of the dynamic system is usually not known and the appropriate attitude to tackle the problem is through Hamilton's law of varying action or, briefly, Hamilton's law

$$\delta\int_{t_0}^{t_f}(T-V)\,\mathrm{d}t+\int_{t_0}^{t_f}\Big(\sum_i f_i\delta q_i\Big)\mathrm{d}t-\sum_i\frac{\partial T}{\partial\dot{q}_i}\delta q_i\Big|_{t_0}^{t_f}=0 \tag{2}$$

In Eq. (2) q_i for the final time t_f are not known and $\delta q_i\neq 0$ for $t=t_f$. Hence, for a correct formulation of Hamilton's law, one must retain the term

$$\sum_i\frac{\partial T}{\partial\dot{q}_i}\delta q_i\Big|_{t_0}^{t_f}$$

In his works, Bailey[12-19] states that for a given dynamic system the initial position q_{0i} and the initial velocities $\dot{q}_{0i}$ are known and therefore cannot be varied. If this is so, Eq. (2) has superflous zeros and it can be rewritten in the following manner:

$$\delta\int_{t_0}^{t_f}(T-V)\,\mathrm{d}t+\int_{t_0}^{t_f}\Big(\sum_i f_i\delta q_i\Big)\mathrm{d}t-\sum_i\frac{\partial T}{\partial\dot{q}_i}\delta q_i(t=t_f);$$
$$\delta q_{0i}=0;\qquad \delta\dot{q}_{0i}=0 \tag{3}$$

Clearly, Eq. (3) looks like a special case of Hamilton's law. It can be shown also that the variational principles for linear dynamics given by Gurtin and others[5-11] are equivalent to Eq. (3) for the linear conservative case. Note, however, that Eq. (3) is not a variational principle even in the linear, conservative case. Is Eq. (3) the only possible case?

For approximate calculations of given dynamic systems, Bailey uses Eq. (3) and the idea of Ritz[20] for the approximation of a function. The admissible function used by Bailey[12-19] and others[23] is "a simple truncated power series." However, our experience is that this choice is not a very good one, confirming the known fact in the literature. The reason is that the matrices obtained by using polynomial series are the notoriously ill-conditioned Hilbert-like matrices. (See, for example, Ref. 21, p. 51 or Ref. 22, pp. 57, 218.) Improvement of the numerical efficiency was obtained by Hitzl[25] who used shifted Legendre polynomials as basis functions. In spite of this, in the numerical examples presented here we also used truncated power series, the reason being that by using them it appears somehow easier to demonstrate the applicability of the different possible formulations of Hamilton's law. However, it must be stated that, in Ref. 4, polynomial segments (that is, finite-element techniques) have been successfully applied to Hamilton's principle, which is, of course, an additional correct special formulation of the general Hamilton's law.

The works of Bailey,[12-19] Smith and Smith,[24] and Simkins[26] have triggered an interesting argument in the literature (see, for example, Refs. 15 and 24-32): How should Hamilton's law be interpreted? Does Hamilton's principle represent a correct formulation for direct calculations of the behavior of a dynamic system and is it a variational principle? The last question seems to be a question of semantics. Smith[31] states that "if a functional exists for which the vanishing of the first variation leads to a physical law then there exists a variational principle." Similar definitions are usually stated in the textbooks of calculus of variations (see, for example, Ref. 33). Hamilton's law, which contains Hamilton's principle as a special case, can be defined in general as a variational statement.[26,32] However, the fact that Hamilton's law and its equivalent, the principle of virtual work, are not variational principles does not preclude them from being useful. The definition of Hamilton's law as a variational statement[26,32] appears to be in line with the definition of the law by Hamilton himself. He called his law "the law of varying action"[1] and stated that "δx, δy, δz are any arbitrary infinitesimal variations of those coordinates."[2] For conservative forces, Hamilton's principle, which is an important case of Hamilton's law, becomes a variational principle[3,24,33] and the functional connected with this principle may or may not have a minimum or maximum value. Then one speaks, in general, of a stationary value of that functional.[3,33,34]

Hamilton's principle has been used to derive dynamic equations since the publication of Hamilton's papers.[1,2] However, the use of Hamilton's principle and the more general Hamilton's law for direct calculations of dynamic systems has begun only recently. In spite of this, the use of numerical methods, based on Hamilton's principle and law, is already growing. Hence, the question of the correctness of Hamilton's principle, raised by Bailey[12-19,30] in connection with its fitness to direct numerical calculations, is more than a question of semantics. In his works,[12-19] Bailey prefers Hamilton's law in the formulation given in Eq. (3). Nevertheless, in Ref. 4 for example, Hamilton's principle [Eq. (1)] was used to develop a space-time finite-element technique for direct calculations of dynamic systems.

It will be shown here that for a one-degree-of-freedom system, there are six and, hence, for an n degree of freedom system 6^n correct formulations of Hamilton's law and all of them can be used for direct computations of dynamic systems.[35]

Six Formulations

Before showing in detail the six different formulations of Hamilton's law, it is important to repeat the basic axiomatic assumption given in Ref. 4 (p. 23): the physical solution of a given dynamic problem exists and it is unique. The word physical means here that, although for some cases there may be more than one mathematical solution, for a real dynamic system, only one of them can be realized physically. It must be stated here that this assumption has been an inherent part of the approach to the solution of almost all problems encountered in engineering. The solution of a problem clearly depends on its initial values. Once we know the existing solution then we will know what the state variables are at any time, including the initial and the final state variables. In other words, there is a one-to-one relationship between the initial and the final values of the problem.

The idea which was proposed in Ref. 4 to make Hamilton's principle usable for numerical calculations will be extended here as follows: we pretend that any possible combination of the initial and final coordinates and velocities of a given dynamic system are known. Hence, their variations are equal to zero. These state variables, although undefined, are treated as being known. After using the proper formulation of Hamilton's law, the actual initial values of the given dynamic problem must be introduced to make the problem well defined. It must be noted again that, by using the basic assumption of definiteness of a given dynamic system, one can use any proper combination of the actual coordinates and velocities at any arbitrary time that has been included in the time region of the given problem as "initial values." However, it will be assumed here that the initial values (for $t=0$) are given. In addition, we will follow the "physical path" for defining the variations of the state variables.[32] In other words, the state coordinates and their variations will be built from the same set of admissible functions. Different arrangements of this basic function set are allowed in order to fulfill the different initial and end requirements of the state variables and their variations.

From the above discussion it is clear that any correct formulation of Hamilton's law must be accompanied by requirements for the initial or final variations of the state variables. For a one-degree-of-freedom dynamic system there exist the six possible combinations shown graphically in Fig. 1.

For simplicity, the time is taken to change between 0 and 1. Following Fig. 1 the six formulations of Hamilton's law for a one-degree-of-freedom system are described below.

F1 Formulation

$$\delta\int_0^1 (T-V)\,\mathrm{d}t+\int_0^1 f\delta q\,\mathrm{d}t-\frac{\partial T}{\partial \dot{q}_1}\delta q_1=0;\ \delta q_0=0;\ \delta\dot{q}_0=0 \qquad (4)$$

The F1 formulation seems to be the most "natural" one: the initial values q_0 and $\dot{q}_0$ are given and their variations vanish. Bailey[12-19] has made intensive use of the F1 formulation (see also Hitzl[25]). As noted above, Gurtin and others[5-11] have developed and used an alternative equivalent formulation to F1.

F2 Formulation

$$\delta\int_0^1 (T-V)\,\mathrm{d}t+\int_0^1 f\delta q\,\mathrm{d}t=0;\ \delta q_0=0;\ \delta q_1=0 \qquad (5)$$

The F2 formulation is in fact the famous Hamilton's principle. For vanishing nonconservative force f, it becomes a variational principle—the functional

$$\int_0^1 (T-V)\,\mathrm{d}t$$

must have a stationary value in the sense that its first variation must vanish.[2,3,33,34] As noted, the F2 formulation was used in Ref. 4 to develop space-time finite elements.

F3 Formulation

$$\delta\int_0^1 (T-V)\,\mathrm{d}t+\int_0^1 f\delta q\,\mathrm{d}t-\frac{\partial T}{\partial \dot{q}}\delta q\Big|_0^1=0;\ \delta\dot{q}_0=0;\ \delta\dot{q}_1=0 \qquad (6)$$

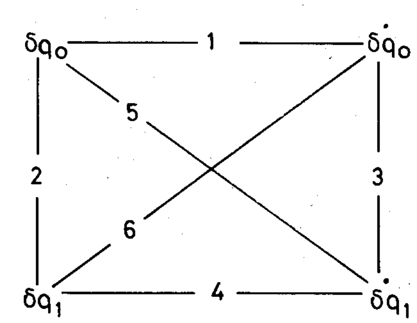

Fig. 1 Six combinations of vanishing variations.

In the F3 formulation Hamilton's law appears in its full form. It can be considered as the counterpart of the F2 formulation.

F4 Formulation

$$\delta\int_0^1 (T-V)\,dt+\int_0^1 f\delta q dt+\frac{\partial T}{\partial \dot{q}_0}\delta q_0=0;\ \delta q_1=0;\ \delta\dot{q}_1=0 \quad (7)$$

It is seen that the F4 formulation can be considered as the counterpart of the F1 formulation.

F5 Formulation

$$\delta\int_0^1 (T-V)\,dt+\int_0^1 f\delta q dt-\frac{\partial T}{\partial \dot{q}_1}\delta q_1=0;\ \delta q_0=0;\ \delta\dot{q}_1=0 \quad (8)$$

In spite of the identical field equation, formulations F1 and F5 are different due to the different requirements of the variations of the state variables.

F6 Formulation

$$\delta\int_0^1 (T-V)\,dt+\int_0^1 f\delta q dt+\frac{\partial T}{\partial \dot{q}_0}\delta q_0=0;\ \delta\dot{q}_0=0;\ \delta q_1=0 \quad (9)$$

Again, due to the different requirements of the variations of the state variables formulations F4 and F6 are different in spite of the identical field equation. The F6 formulation can be considered as the counterpart of the F5 formulation.

Note that the initial and final variations of the velocities do not appear in the general formulation of Hamilton's law, [Eq. (2)]. However, following the "physical path" already stated, these variations were added here to make the formulations well defined.

Numerical Examples

All six formulations of Hamilton's law given in the previous section can be used for direct calculations of a given dynamic system. In order to demonstrate one of the techniques applied to the above formulations, we have purposely chosen a very simple example: on a mass of magnitude one acts a time-dependent force of magnitude $6t$. The actual initial values of the state variables are zero and one, respectively. For this case one obtains the following values:

$$T=\tfrac{1}{2}\dot{q}^2;\ V=0;\ f=6t;\ q_0=0\ \ \dot{q}_0=1;\ 0\le t\le 1 \quad (10)$$

Clearly, the exact solution of Eq. (10) is

$$q=t+t^3 \quad (11)$$

For the case given in Eq. (10) the general formulation of Hamilton's law [Eq. (2)], obtains the following form

$$\int_0^1 (\dot{q}\delta\dot{q}+6t\delta q)\,dt-\dot{q}\delta q\Big|_0^1=0 \quad (12)$$

Now, we will try to approximate the supposedly unknown solution of Eq. (10) by the following polynomial series

$$\bar{q}=a_0+a_1t+a_2t^2 \quad (13)$$

For all six formulations the approximate "actual" coordinate $\bar{q}$ must fulfill the initial actual conditions and then

$$\bar{q}=t+a_2t^2 \qquad \dot{\bar{q}}=1+2a_2t \quad (14)$$

For every different formulation the variations $\delta\bar{q}$ and $\delta\dot{\bar{q}}$ have to fulfill different requirements. These requirements will be satisfied by arranging Eq. (13) in a different way for each case.

F1 formulation:

$$\bar{q}=\bar{q}_0+\dot{\bar{q}}_0t+a_2t^2;\ \delta\bar{q}_0=0;\ \delta\dot{\bar{q}}_0=0$$
$$\delta\bar{q}=t^2\delta a_2;\quad \delta\dot{\bar{q}}=2t\delta a_2 \quad (15)$$

Substitution of Eqs. (14) and (15) into Eq. (4) yields

$$[3/2-(2/3)a_2]\delta a_2=0;\quad a_2=9/4$$
$$\bar{q}=t+(9/4)t^2 \quad (16)$$

F2 formulation:

$$\bar{q}=\bar{q}_0(1-t)+\bar{q}_1t+a_2(t^2-t);\ \delta\bar{q}_0=0;\ \delta\bar{q}_1=0$$
$$\delta\bar{q}=(t^2-t)\delta a_2;\quad \delta\dot{\bar{q}}=(2t-1)\delta a_2 \quad (17)$$

Substitution of Eqs. (17) and (14) into Eq. (5) yields

$$[(1/3)a_2-1/2]\delta a_2=0;\quad a_2=3/2$$
$$\bar{q}=t+(3/2)t^2 \quad (18)$$

Here, the force f can be derived from a potential field and the F2 case has a functional I

$$I=\int_0^1 (T-V)\,dt \qquad \bar{T}=\tfrac{1}{2}\dot{\bar{q}}^2$$

$$\bar{V}=-6t\bar{q} \qquad \bar{I}=\int_0^1 (\tfrac{1}{2}\dot{\bar{q}}^2+6t\bar{q})\,dt \quad (19)$$

Substitution of Eq. (17) into Eq. (19) yields

$$\bar{I}=\tfrac{1}{2}(\bar{q}_1-\bar{q}_0)^2+\bar{q}_0+2\bar{q}_1+(1/6)a_2^2-\tfrac{1}{2}a_2 \quad (20)$$

In Eq. (20) $\bar{q}_0$ and $\bar{q}_1$ are supposedly known and the results of the partial differentiation of $\bar{I}$ with respect to them does not necessarily vanish. In Eq. (20) only a_2 varies and only it can be used to make $\bar{I}$ stationary,

$$\frac{\partial\bar{I}}{\partial a_2}=\frac{1}{3}a_2-\frac{1}{2}=0;\ a_2=\frac{3}{2} \quad (21)$$

Of course, the approximate solution obtained from Eq. (21) is identical with the one given in Eq. (18).

F3 formulation:

$$\bar{q}=a_0+\dot{\bar{q}}_0(t-\tfrac{1}{2}t^2)+\tfrac{1}{2}\dot{\bar{q}}_1t^2;\ \delta\dot{\bar{q}}_0=0;\ \delta\dot{\bar{q}}_1=0$$
$$\delta\bar{q}=\delta a_0;\quad \delta\dot{\bar{q}}=0 \quad (22)$$

From Eqs. (22), (14), and (6),

$$(3-2a_2)\delta a_0=0;\quad a_2=3/2$$
$$\bar{q}=t+(3/2)t^2 \quad (23)$$

F4 formulation:

$$\bar{q}=\bar{q}_1+\dot{\bar{q}}_1(t-1)+(t-1)^2a_2;\ \delta\bar{q}_1=0;\ \delta\dot{\bar{q}}_1=0$$
$$\delta\bar{q}=(t-1)^2\delta a_2$$
$$\delta\dot{\bar{q}}=2(t-1)\delta a_2 \quad (24)$$

From Eqs. (24), (14), and (7),

$$[1/2-(2/3)a_2]\delta a_2=0; \quad a_2=3/4$$

$$\bar{q}=t+(3/4)t^2 \tag{25}$$

F5 formulation:

$$\bar{q}=\bar{q}_0+\dot{\bar{q}}_1 t+(t^2-2t)a_2; \; \delta\bar{q}_0=0; \; \delta\dot{\bar{q}}_1=0$$

$$\delta\bar{q}=(t^2-2t)\delta a_2$$

$$\delta\dot{\bar{q}}=2(t-1)\delta a_2 \tag{26}$$

From Eqs. (26), (14), and (8),

$$[(4/3)a_2-5/2]\delta a_2=0; \quad a_2=15/8$$

$$\bar{q}=t+(15/8)t^2 \tag{27}$$

F6 formulation:

$$\bar{q}=\bar{q}_1+\dot{\bar{q}}_0(t-1)+(t^2-1)a_2; \; \delta\dot{\bar{q}}_0=0; \; \delta\bar{q}_1=0$$

$$\delta\bar{q}=(t^2-1)\delta a_2; \qquad \delta\dot{\bar{q}}=2t\delta a_2 \tag{28}$$

From Eqs. (28), (14), and (9),

$$[(4/3)a_2-3/2]\delta a_2=0; \quad a_2=9/8$$

$$\bar{q}=t+(9/8)t^2 \tag{29}$$

It is interesting to compare the different approximate solutions with the exact one. From Fig. 2 one can see that the best approximation for this case is obtained by the F4 formulation and the worst one for the F1 formulation. To make the comparison more quantitative, an effective coefficient E_f will be defined as follows:

$$E_f=1-\left[\int_0^1 (q-\bar{q})^2 \mathrm{d}t\right]^{1/2} \Big/ \left[\int_0^1 q^2 \mathrm{d}t\right]^{1/2} \tag{30}$$

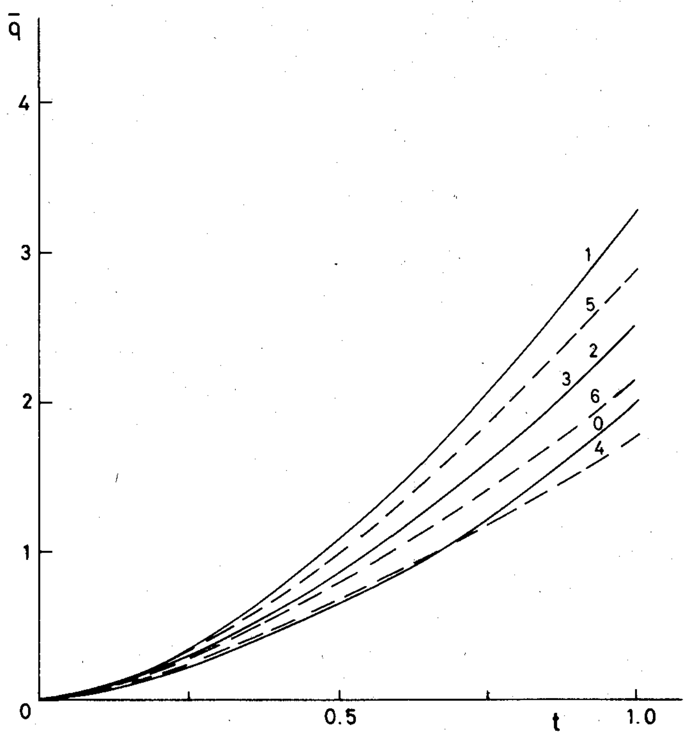

Fig. 2 Approximate solutions 1÷6 compared with the exact solution (0).

where q and $\bar{q}$ are the exact and approximate solutions, respectively.

From Table 1 it is clear, once again, that for the simple special problem treated here the F4 and the F1 formulations of Hamilton's law provide the best and the worst approximations, respectively. However, it must be noted that no general conclusions for preference of any one of the formulations can be derived from this simple example. It seems that for every special problem one of the formulations would be the "best."

When more terms were added to the polynomial series [Eq. (13)], every one of the six formulations provided the following matrix equation:

$$Ma=p \qquad a^T=[a_2,a_3...a_N] \tag{31}$$

where the matrix M and the vector p were different for the different formulations. In spite of this, the solution of Eq. (31) for $N>2$ obtained from all of the six formulations is

$$a_3=1$$

$$a_j=0; \quad j=2, \; 4\div N \tag{32}$$

which is, of course, the exact solution of the problem. This was expected because the trial functions included the exact solution. It can be seen that all six formulations of Hamilton's law provided the exact solution in their own way.

Another numerical example was a free mass-spring system with one degree of freedom and mass and spring coefficient equal to unity. For this system the general Hamilton's law reads

$$\delta\int_0^k (\tfrac{1}{2}\dot{q}^2-\tfrac{1}{2}q^2)\mathrm{d}t-\dot{q}\delta q\Big|_0^k=0$$

$$q_0=0; \quad \dot{q}_0=1$$

$$0\leq t\leq k; \quad k=4.5\pi \tag{33}$$

The time interval was taken to be 4.5π so that at the end of the interval the displacement is different from zero. Note that in the exact solution the velocity vanishes at the end of the interval. However, this fact was not used as a condition in the approximate solution.

A simple transformation makes the numerical solution easier,

$$t=k\tau \tag{34}$$

by substitution of Eq. (34) into Eq. (33) one obtains

$$\int_0^1 \left[\frac{\mathrm{d}q}{\mathrm{d}\tau}\delta\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)-k^2 q\delta q\right]\mathrm{d}\tau-\frac{\mathrm{d}q}{\mathrm{d}\tau}\delta q\Big|_0^1=0$$

$$q_0=0; \quad \frac{\mathrm{d}q_0}{\mathrm{d}\tau}=k \text{ for } \tau=0 \tag{35}$$

The assumed solution follows.

The actual approximate solution for all six formulations is

$$\bar{q}=k\tau+\sum_{i=2}^{N}\tau^i a_i \tag{36}$$

Table 1 Effective coefficients for mass-force system, $N=2$

Formulation	E_f	Formulation	E_f
F1	0.320	F4	0.922
F2	0.675	F5	0.500
F3	0.675	F6	0.845

Table 2 Effective coefficients for mass-spring system

	N			
Formulation	5	10	15	20
F1	0.213672	0.654571	0.999147	1.00000
F2	0.124141	0.979067	0.999924	1.00000
F3	−0.237049	0.991671	0.999986	1.00000
F4	−0.914004	0.993651	0.999853	1.00000
F5	0.108923	0.975620	0.999210	1.00000
F6	−0.193683	0.992793	0.999924	1.00000

where the virtual displacement for the six different formulations are

$$\text{F1:}\quad \delta\bar{q}=\sum_{j=2}^{N}\tau^{j}\delta a_j$$

$$\text{F2:}\quad \delta\bar{q}=\sum_{j=2}^{N}(\tau^{j}-\tau)\delta a_j$$

$$\text{F3:}\quad \delta\bar{q}=\delta a_0+\sum_{j=3}^{N}(\tau^{j}-0.5j\tau^2)\delta a_j$$

$$\text{F4:}\quad \delta\bar{q}=\sum_{j=2}^{N}(\tau^{j}-j\tau+j-1)\delta a_j$$

$$\text{F5:}\quad \delta\bar{q}=\sum_{j=2}^{N}(\tau^{j}-j\tau)\delta a_j$$

$$\text{F6:}\quad \delta\bar{q}=\sum_{j=2}^{N}(\tau^{j}-1)\delta a_j \tag{37}$$

The exact solution of the mass-spring system treated here is simply

$$q=\sin t \tag{38}$$

Clearly the exact solution [Eq. (38)] is not included in the assumed approximate solution [Eq. (36)]. Note that the same actual approximate and virtual displacements [Eqs. (36) and (37)] were used in the first example.

Substitution of Eqs. (36) and (37) into Eq. (35) yields equations of the type given in Eq. (31). As stated before, the matrix M obtained by using simple polynomial series is a Hilbert-type ill-conditioned matrix. The solution of Eq. (31) was obtained by using quadruple precision on an IBM 370/168 VS2 computer. In this way, an excellent convergence was obtained with $N=20$ for all formulations. However, one can see in Table 2 that each of the six formulations converges in its own way.

Note that the "natural" F1 formulation is again not the best one. Several cases with different time intervals (for example, $k=4.25\pi$ and 4.71π) were calculated using the same approach. The results obtained were very similar to the ones given in Table 2.

Discussion

The numerical solutions can be improved by using better approximate functions. In the authors' opinion the finite-element technique provides an excellent method for the automatic selection of the approximate functions. In Ref. 4 the finite-element technique was applied to the F2 formulation of Hamilton's law. However, some additional analysis have shown that each of the six formulations described here can be used to develop a time-space finite-element technique.

By performing the variation and integration of Eq. (2) one obtains,

$$\int_{t_0}^{t_1}\sum_i\left[\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_i}-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}-f_i\right]\delta q_i dt=0 \tag{39}$$

where in the integrand appear the famous Lagrange equations of the second kind.[36] Equation (39) is the principle of virtual work. Many textbooks take Eq. (39) as justification of Hamilton's law (see, for example, Ref. 3, pp. 90-95). Note that in comparison with Eq. (39), Hamilton's law [Eq. (2)] takes full advantage of the existence of kinetic and potential energy.

Once again, every one of the six formulations will make Eq. (39) well defined. However, Eq. (39) shows a further possible separation between the actual approximate displacements and their variations[31] and more possible formulations. In other words, one can build q and δq from different sets of functions. This possibility can be used to improve the accuracy and the numerical stability of the finite-element technique. Some of the preliminary results obtained by the authors using different functions for q and δq are encouraging.

Conclusions

It was shown and numerically demonstrated that there exist 6^n correct formulations of Hamilton's law of varying action. It is the authors' hope that this paper will aid in resolving the controversy connected with this law. There are many possible formulations which, when properly applied, can be used for direct calculations of the behavior of dynamic systems.

Acknowledgment

This paper is based partly on the Doctoral Thesis of the second author. It is to be submitted to the Senate of the Technion—Israel Institute of Technology.

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